Academic Open Mathematics Research Journal
Vol. 1, No. 1, January 2016, pp. 1-13
Available online at http://www.akrpub.com/Journals.php

**Research article**

# A Stone-type extension for non-additive set functions

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## Abstract

We consider some basic properties of the disjoint variation of lattice group-valued set functions. Moreover, using the Maeda-Ogasawara-Vulikh representation theorem, we prove an extension result for $k$-subadditive lattice group-valued capacities, in which $(s)$-boundedness, continuity from above and from below are intended in the classical like sense, and not necessarily with respect to a single order sequence or regulator. Furthermore we pose some open problems.

**Keywords:** lattice group, capacity, $k$-subadditive set function, continuous set function, disjoint variation, semivariation.

**2010 A.M.S. Subject Classifications:** 28A05, 28B10, 28B15, 46G10.

## 1.Introduction

In the literature there are many studies about non-additive set functions. For an overview, we quote for example [19,29,33] and their bibliographies. These topics have several applications, for instance to intuitionistic fuzzy events,

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fuzzy measures, belief functions and observables (see also [1,30,33]) and to decision theory and mathematical economics (see also [2]).
In this paper we deal with extension results for lattice group-valued $k$-subadditive capacities from an algebra to the least $\sigma$-algebra containing it. We extend earlier results proved in [7,9,20,29]. Note that in [35] it is shown that a $\sigma$-additive set function, defined on any algebra $\mathcal{W}$ of any abstract infinite set $G$ and taking valued in a Dedekind complete vector lattice $R$, has an extension to the $\sigma$-algebra $\sigma(\mathcal{W})$ generated by $\mathcal{W}$ if and only if $R$ is weakly $\sigma$-distributive. However, if $R$ is not weakly $\sigma$-distributive, it is still possible to have such an extension, if the involved algebra $\mathcal{W}$ satisfies suitable properties. This is the case of *perfect* algebras, like for instance the algebra of all open-closed subsets of a compact totally disconnected topological space (see also [9,12,15,28]), which is widely used in the literature to construct Stone-type extensions, which have several applications, for example to convergence and decomposition theorems (see also [10,11,12,18,31,32]). We use the Maeda-Ogasawara-Vulikh representation theorem for lattice groups (see also [8]), the extension results proved in [20] in the non-additive setting, and deduce the required properties of the extension by the corresponding ones of the associated real-valued extensions. In our setting, in general the involved set functions are not finitely additive, and hence we consider the *disjoint variation* (see also [29,36,37]), by means of which it is possible to overcome technical difficulties and to obtain $(s)$-boundedness of the "components" of the involved set functions, from which it is possible to deduce $(s)$-boundedness of our starting set function. This fact, in the finitely additive case, is guaranteed by the only boundedness. We also show that our approach *strictly* includes the finitely additive case. Observe that in our context, $(s)$-boundedness, continuity from above and from below are intended in the classical like sense, and not necessarily with respect to a single order sequence or regulator.
Some other results about extensions of finitely additive or modular measures and related topics can be found, for instance, in [3,4,5,6,9,15,16,22,23,24,25,26,27,28,34,35]. Finally, we pose some open problems.

## 2.Method and tools

It is dealt with the problem of extending a non-additive lattice group-valued set function, satisfying suitable properties, from a perfect algebra to the smallest $\sigma$-algebra containing it. In general, if the involved algebra is arbitrary and the considered lattice group is not weakly $\sigma$-distributive, this problem has no solution (see also [35]). Since we treat the non-additive case, in order to overcome the technical difficulties we consider $k$-subadditive capacities of bounded disjoint variation, and we show that our setting includes strictly the finitely additive case. We prove that, differently to finitely additive case, boundedness of a $k$-subadditive capacity does not imply that it is of bounded disjoint variation, and this property is not a necessary condition for $(s)$-boundedness. In order to prove our main results, we use the analogous ones proved for real-valued capacities (see also [7,20]) and the Maeda-Ogasawara-Vulikh representation theorem of lattice groups as suitable subgroups of continuous functions, by means of which it is possible to prove some properties of lattice group-valued set functions by using the corresponding ones of the real-valued set functions and by a density argument, which uses the Baire category theorem. We use perfectness of the starting algebra and boundedness of the disjoint variation to prove that the "components" of the given set functions are continuous from above and from below and $(s)$-bounded, respectively. This is crucial, in order to apply the extension results existing in the literature in the real case and the Maeda-Ogasawara-Vulikh theorems.

## 3.Preliminaries

We begin with recalling the following basic concepts on lattice groups (see also [12]).

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**Definitions 3.1.**

(a) Let $R$ be a Dedekind complete lattice group. A sequence $(\sigma_p)_p$ in $R$ is called $(O)$-*sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$.

(b) A sequence $(x_n)_n$ in $R$ is said to be *order convergent* (or $(O)$-convergent ) to $x$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that for every $p \in \mathbb{N}$ there is a positive integer $n_0$ with $|x_n - x| \leq \sigma_p$ for each $n \geq n_0$, and in this case we write $(O) \lim_n x_n = x$.

(c) If $(x_n)_n$ is a bounded sequence in $R$, then set

$$\limsup_n x_n = \bigwedge_{s=1}^{\infty} ( \bigvee_{n=s}^{\infty} x_n), \quad \liminf_n x_n = \bigvee_{s=1}^{\infty} ( \bigwedge_{n=s}^{\infty} x_n).$$

Note that $(O) \lim_n x_n = x$ if and only if $\limsup_n x_n = \liminf_n x_n = x$ (see also [12]).

(d) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in $R$ the limit $(O) \lim_n \sum_{j=1}^{n} x_j$, if it exists in $R$.

We now recall the Maeda-Ogasawara-Vulikh theorem, which gives a representation of lattice groups as subsets of continuous extended real-valued functions defined on suitable topological spaces (see also [8,12]). From now on, we denote by the symbols $\vee$ and $\wedge$ the supremum and infimum in $R$ and by sup and inf the pointwise supremum and infimum or the supremum and infimum in $\mathbb{R}$, respectively.

**Theorem 3.2.** *Given a Dedekind complete lattice group $R$, there exists a compact extremely disconnected topological space $\Omega$, unique up to homeomorphisms, such that $R$ can be embedded isomorphically as a subgroup of $C_{\infty}(\Omega) = \{f \in \widetilde{\mathbb{R}}^{\Omega} : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if we denote by $\hat{a}$ an element of $C_{\infty}(\Omega)$ which corresponds to $a \in R$ under the above isomorphism, then for any family $(a_\lambda)_{\lambda \in \Lambda}$ of elements of $R$ with $R \ni a_0 = \bigvee_{\lambda \in \Lambda} a_\lambda$ (where the supremum is taken with respect to $R$), then $\widehat{a_0} = \bigvee_{\lambda \in \Lambda} \widehat{a_\lambda}$ with respect to $C_{\infty}(\Omega)$, and we get $\widehat{a_0}(\omega) = sup_\lambda \widehat{a_\lambda}(\omega)$ in the complement of a meager subset of $\Omega$. The same is true for $\bigwedge_{\lambda \in \Lambda} a_\lambda$.*

We now recall some fundamental properties of lattice group-valued capacities (see also [13,17,29]). From now on, $R$ is any Dedekind complete lattice group, $G$ is any infinite set, $\mathcal{P}(G)$ is the family of all subsets of $G$, $\Sigma \subset \mathcal{P}(G)$ is a $\sigma$-algebra, $\mathcal{W} \subset \mathcal{P}(G)$ is an algebra, $\sigma(\mathcal{W})$ is the smallest sub-$\sigma$-algebra of $\mathcal{P}(G)$ containing $\mathcal{W}$, $k$ is a fixed positive integer, and, when it is not said explicitly, $m: \mathcal{W} \to R$ or $m: \Sigma \to R$ is a positive set function.

**Definitions 3.3.** (a) We say that $m: \mathcal{W} \to R$ is *$k$-subadditive* on $\mathcal{W}$ iff

$$m(A \cup B) \leq m(A) + k\, m(B) \quad \text{whenever } A, B \in \mathcal{W},\ A \cap B = \emptyset. \tag{1}$$

(b) A *capacity* $m: \mathcal{W} \to R$ is an increasing set function with $m(\emptyset) = 0$.

(c) When $R = \mathbb{R}$, a 1-subadditive capacity is called also a *submeasure* (see also [12,20]).

**Remark 3.4.** Observe that, if $m: \mathcal{W} \to R$ is a $k$-subadditive capacity, then $m(A \cup B) \leq m(A) + k\, m(B)$ for *any* $A$, $B \in \Sigma$. Indeed, thanks to monotonicity and $k$-subadditivity, we have

$$m(A \cup B) = m(A \cup (B \setminus A)) \leq m(A) + k\, m(B \setminus A) \leq m(A) + k\, m(B)$$

whenever $A, B \in \Sigma$.

**Definitions 3.5.** (a) A positive set function $m: \mathcal{W} \to R$ is *continuous from above at* $\emptyset$ *on* $\mathcal{W}$ iff

$$\bigwedge_{n=1}^{\infty} m\,(E_n) = (O) \lim_n m\,(E_n) = 0$$

for every decreasing sequence $(E_n)_n$ in $\mathcal{W}$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$.

(b) We say that $m$ is *continuous from above* (*from below*, respectively) *on* $\mathcal{W}$ iff

$$\bigwedge_{n=1}^{\infty} m\,(E_n) = (O) \lim_n m\,(E_n) = m(E)$$

$$\left( \bigvee_{n=1}^{\infty} m\,(E_n) = (O) \lim_n m\,(E_n) = m(E), \text{ respectively } \right)$$

for every decreasing (increasing, respectively) sequence $(E_n)_n$ in $\mathcal{W}$ with $\bigcap_{n=1}^{\infty} E_n = E \in \mathcal{W}$ $\left( \bigcup_{n=1}^{\infty} E_n = E \in \mathcal{W}\right.$, respectively).

(c) We say that $m$ is $(s)$-*bounded on* $\mathcal{W}$ iff $(O) \lim_n m\,(C_n) = 0$ for every disjoint sequence $(C_n)_n$ in $\mathcal{W}$.

The following result holds.

**Proposition 3.6.** *Let* $m: \mathcal{W} \to R$ *be a* $k$*-subadditive capacity, continuous from above at* $\emptyset$*. Then* $m$ *is continuous from above and from below.*

**Proof.** We first prove continuity from above. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{W}$, $A := \bigcap_{n=1}^{\infty} A_n$, $A \in \mathcal{W}$, and let $B_n := A_n \setminus A$. We get $B_n \in \mathcal{W}$ for each $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and hence

$$(O) \lim_n m\,(B_n) = \bigwedge_{n=1}^{\infty} m\,(B_n) = 0.$$

Taking into account monotonicity and $k$-subadditivity of $m$, we obtain

$$0 \le m(A_n) - m(A) \le k\, m(A_n \setminus A) = k\, m(B_n),$$

and so

$$0 \le \limsup_n (\, m(A_n) - m(A)) \le k \bigwedge_{n=1}^{\infty} m\,(B_n) = 0.$$

Therefore $(O) \lim_n (\, m(A_n) - m(A)) = 0$, namely $(O) \lim_n (\, m(A_n)) = m(A)$, that is

$$m(A) = (O) \lim_n m\,(A_n) = \bigwedge_{n=1}^{\infty} m\,(A_n).$$

Thus, we obtain continuity from above of $m$.

We now prove continuity from below. Let $(E_n)_n$ be an increasing sequence of elements of $\mathcal{W}$, $E := \bigcup_{n=1}^{\infty} E_n$, $E \in \mathcal{W}$. Let $F_n := E \setminus E_n$, $n \in \mathbb{N}$. Note that $F_n \in \mathcal{W}$ for every $n \in \mathbb{N}$ and that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Hence, by hypothesis, we get $(O)\lim_n m(F_n) = \bigwedge_{n=1}^{\infty} m(F_n) = 0$. By monotonicity and $k$-subadditivity of $m$, we have

$$0 \le m(E) - m(E_n) \le k\, m(E \setminus E_n) = k\, m(F_n),$$

and hence

$$0 \le \limsup_n (m(E) - m(E_n)) \le k \bigwedge_{n=1}^{\infty} m(F_n) = 0.$$

Thus, $(O)\lim_n (m(E) - m(E_n)) = 0$, that is $m(E) = (O)\lim_n m(E_n) = \bigvee_{n=1}^{\infty} m(E_n)$. So we get that $m$ is continuous from below. □

If $m: \mathcal{W} \to R$ ($m: \Sigma \to R$, respectively), we denote by

$$v(m)(A) := \bigvee \{m(B): B \in \mathcal{W}, B \subset A\}, A \in \mathcal{W}$$
$$(v(m)(A) := \bigvee \{m(B): B \in \Sigma, B \subset A\}, A \in \Sigma \text{ respectively})$$

the *semivariation* of $m$. Note that the semivariation satisfies the following property, which for a sake of simplicity we give only with respect to $\mathcal{W}$, but which holds also with respect to $\Sigma$.

**Proposition 3.7.** *If $m: \mathcal{W} \to R$ is $k$-subadditive, then $v(m)$ is $k$-subadditive too.*

**Proof.** Choose arbitrarily $A, B \in \mathcal{W}$ with $A \cap B = \emptyset$, pick $C \in \mathcal{W}$ with $C \subset A \cup B$ and set $C_1 := A \cap C$, $C_2 := B \cap C$. Note that $C_1 \cap C_2 = \emptyset$. By (1) we get $m(C) \le m(C_1) + k\, m(C_2)$, and hence $m(C) \le v(m)(A) + k\, v(m)(B)$. By arbitrariness of $C$ we get $k$-subadditivity of $v(m)$. □

**Remark 3.8.** Observe that, if $m: \Sigma \to R$ is a $k$-subadditive capacity, continuous from above at $\emptyset$, then $m$ is $(s)$-bounded. Indeed, if $(C_h)_h$ is any disjoint sequence in $\Sigma$, then for every $h \in \mathbb{N}$ we get

$$0 \le m(C_h) \le m(\bigcup_{i=h}^{\infty} C_i) + k\, m(\bigcup_{i=h+1}^{\infty} C_i).$$

Taking the $(O)$-limit as $h$ tends to $+\infty$, by virtue of monotonicity, $k$-subadditivity and continuity from above at $\emptyset$ we get $(O)\lim_h m(C_h) = 0$. By arbitrariness of the chosen sequence $(C_h)_h$ we get $(s)$-boundedness of $m$. □

We now give the following technical proposition.

**Proposition 3.9.** *Let $m: \mathcal{W} \to R$ be a bounded positive set function and $\Omega$ be as in Theorem* 3.2. *If there is a meager set $N_* \subset \Omega$ such that the set functions $m(\cdot)(\omega)$ are real-valued and $k$-subadditive capacities for each $\omega \in \Omega \setminus N_*$, then $m$ is a $k$-subadditive capacity. Moreover, if $m$ is a $k$-subadditive capacity, then the set functions $m(\cdot)(\omega)$ are $k$-subadditive real-valued capacities for each $\omega \in \Omega$.*

**Proof.** First of all observe that, since $m$ is bounded, then the range of $m$ is embedded in the space $\mathcal{C}(\Omega) := \{f: \Omega \to \mathbb{R}, f \text{ is continuous}\}$, where $\Omega$ is as in Theorem 2 (see also [21, Theorem 4.1], [35, p. 69]). Hence, for every $\omega \in \Omega$ the set function $m_\omega$ defined by $m_\omega(A) := m(A)(\omega)$, $A \in \mathcal{W}$, is real-valued. We now prove the first part. If $N_*$ is as in the hypothesis, then

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Available online at http://www.akrpub.com/Journals.php

$$m(A \cup B)(\omega) \leq m(A)(\omega) + k\, m(B)(\omega),$$
$$0 = m(\emptyset)(\omega) \leq m(A)(\omega) \leq m(B)(\omega)$$

for any $A, B \in \mathcal{W}$ and $\omega \in \Omega \setminus N_*$. Since $N_*$ is meager, thanks to Theorem 3.2, by a density argument it follows that

$$m(A \cup B) \leq m(A) + k\, m(B),$$
$$0 = m(\emptyset) \leq m(A) \leq m(B)$$

for every $A, B \in \mathcal{W}$, that is $m$ is a $k$-subadditive capacity. The proof of the last part is analogous, by reversing the argument. □

## 4.The main results

We begin with treating $(s)$-boundedness of $k$-subadditive capacities. In general, differently from the finitely additive setting, it is not true that every bounded $k$-subadditive capacity is $(s)$-bounded, even in the real case. For example let $G = [1,2]$, $\mathcal{W} = \mathcal{P}(G)$, and set

$$m(\emptyset) = 0 \text{ and } m(A) = \sup\, A \tag{2}$$

if $A \subset G$, $A \neq \emptyset$. It is not difficult to see that $m$ is positive, monotone and 1-subadditive. For any disjoint sequence $(A_n)_n$ of nonempty subsets of $G$ we get $m(A_n) \geq 1$ for every $n \in \mathbb{N}$, and so it is not true that $\lim_n m\,(A_n) = 0$. Thus $m$ is not $(s)$-bounded.

So, we consider the *disjoint variation* of a lattice group-valued set function (see also [29,36]).

**Definitions 4.1.** (a) Let us add to $R$ an extra element $+\infty$, obeying to the usual rules, and for a positive set function $m\colon \mathcal{W} \to R$ define the *disjoint variation* $\overline{m}\colon \mathcal{W} \to R \cup \{+\infty\}$ of $m$ by

$$\overline{m}(A) := \bigvee_I \Big(\sum_{i \in I} m\,(D_i)\Big), \quad A \in \mathcal{W}, \tag{3}$$

where the supremum in (3) is intended with respect to all finite disjoint families $\{D_i : i \in I\}$ with $D_i \in \mathcal{W}$ and $D_i \subset A$ for each $i \in I$.

(b) A set function $m$ is said to be *of bounded disjoint variation* on $\mathcal{W}$ (shortly, $BDV$) iff $\overline{m}(G) \in R$, where $\overline{m}$ is as in (3).

**Examples 4.2.** (a) Let $m$ be as in (2), it is not difficult to see that $v(m)(G) = 2$. Fix arbitrarily $n \in \mathbb{N}$, and set $D_i = [1 + \frac{i-1}{n}, 1 + \frac{i}{n}[$, $i = 1, \ldots, n$. We get $m(D_i) = \sup\, D_i \geq 1$, and hence $\sum_{i=1}^{n} m\,(D_i) \geq n$. From this and by arbitrariness of $n$ it follows that $\overline{m}(G) = +\infty$, and so $m$ is not $BDV$. Hence, in general boundedness does not imply $BDV$, though it is readily seen that the converse implication is true.

(b) Let $m_0\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ be defined by $m_0(A) := \sum_{n \in A} \frac{(-1)^n}{n^2}$, $A \subset \mathbb{N}$, and set $m^*(A) := |m_0(A)|$,

$$m(A) := v(m^*)(A) = \sup\{\, |m_0(B)| : B \subset A\} = \sup\Big\{\Big|\sum_{n \in B} \frac{(-1)^n}{n^2}\Big| : B \subset A\Big\},\ A \subset \mathbb{N}.$$

Note that $m^*$ is not increasing, since $m^*(\{1,3\}) = \frac{10}{9} > \frac{31}{36} = m^*(\{1,2,3\})$. It is not difficult to check that $m^*$ is 1-subadditive on $\mathcal{P}(\mathbb{N})$. Hence, by Proposition 3.7, $m$ is 1-subadditive on $\mathcal{P}(\mathbb{N})$ too. Moreover, by construction, $m$ is positive and monotone, and $m(\emptyset) = 0$. Furthermore we get

$$0 \leq \overline{m}(\mathbb{N}) = \sup_I (\sum_{i\in I} m\,(D_i)) = \sup_I (\sum_{i\in I} (\max_{B\subset D_i} |\sum_{n\in B} \frac{(-1)^n}{n^2}|)) \leq$$

$$\leq \sup_I (\sum_{i\in I} (\sum_{n\in D_i} \frac{1}{n^2})) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad (4)$$

where the involved supremum is taken with respect to all finite disjoint families $\{D_i : i \in I\}$ such that $D_i \subset \mathbb{N}$ for every $i \in I$, and hence $m$ is $BDV$. Note that the supremum in (4) is exactly equal to $\frac{\pi}{6}$: indeed it is enough to consider, for each $n \in \mathbb{N}$, the family $\{D_j := \{j\}: j = 1, \dots, n\}$ and to take into account that $m(\{j\}) = \frac{1}{j^2}$ for every $j \in \mathbb{N}$. Finally, we have

$$m(\{1,2\}) = \max\{1, \frac{1}{4}, \frac{3}{4}\} = 1 < \frac{5}{4} = 1 + \frac{1}{4} = m(\{1\}) + m(\{2\}),$$

and so $m$ is not finitely additive.

(c) Let $G = [1,1]$, $\Sigma$ be the $\sigma$-algebra of all Borel subsets of $G$, $m_0(A) = \int_A \mathrm{s\,gn}\ x\,dx$, $A \in \Sigma$, where $\mathrm{sgn}(x) = 1$ if $x \in ]0,1]$, $\mathrm{sgn}(x) = -1$ if $x \in [-1,0[$ and $\mathrm{sgn}(0) = 0$, and put $m^*(A) = \sqrt{|m_0(A)|}$, $A \in \Sigma$ (see also [29, Example 3.1]). Note that $m^*$ is not monotone: indeed we have

$$m^*(G) = \sqrt{|m_0(G)|} = \sqrt{|\int_{-1}^{1} \mathrm{s\,gn}\ x\,dx|} = 0 = m^*(\emptyset),$$

$$m^*([0,1]) = \sqrt{|m_0([0,1])|} = \sqrt{|\int_{0}^{1} \mathrm{s\,gn}\ x\,dx|} = 1.$$

Now, fix arbitrarily $n \in \mathbb{N}$ and pick $D_i = [\frac{i-1}{n}, \frac{i}{n}[$, $i = -n+1, -n+2, \dots, -1, 0, 1, \dots, n$. We have

$$\overline{m^*}(G) \geq \sum_{i=-n+1}^{n} \sqrt{\frac{1}{n}} = \frac{2\,n}{\sqrt{n}} = 2\sqrt{n}.$$

From this, by arbitrariness of $n$, it follows that $m^*$ is not $BDV$.

We prove that $m^*$ is 1-subadditive. Fix arbitrarily two disjoint sets $A, B \in \Sigma$. We get

$$m^*(A \cup B) = \sqrt{|m_0(A \cup B)|} = \sqrt{|\int_{A\cup B} \mathrm{s\,gn}\ x\,dx|} =$$

$$= \sqrt{|\int_A \mathrm{s\,gn}\ x\,dx + \int_B \mathrm{s\,gn}\ x\,dx|} \leq \sqrt{|\int_A \mathrm{s\,gn}\ x\,dx| + |\int_B \mathrm{s\,gn}\ x\,dx|} =$$

$$= \sqrt{|m_0(A)| + |m_0(B)|} \leq \sqrt{|m_0(A)|} + \sqrt{|m_0(B)|} = m^*(A) + m^*(B).$$

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Set now $m(A) := v(m^*)(A) = \sup\{m^*(B): B \in \Sigma, B \subset A\}$, $A \in \Sigma$. Of course, $m$ is positive and increasing. Since $m^*$ is not $BDV$, then a fortiori $m$ is not. Moreover, by Proposition 3.7, $m$ is 1-subadditive, since $m^*$ is. Furthermore, it is not difficult to see that $m^*$ is $(s)$-bounded, and hence, by [37, Theorem 2.2], $m$ is $(s)$-bounded too. Thus property $BDV$ is not a necessary condition for $(s)$-boundedness of $k$-subadditive capacities. □

We now give a sufficient condition for $(s)$-boundedness of a set function $m$ with values in a lattice group $R$ and of its "components" $m(\cdot)(\omega)$, when $R$ is considered as a subgroup of $\mathcal{C}_\infty(\Omega)$ as in Theorem 3.2, for $\omega$ belonging to the complement of a suitable meager subset of $\Omega$.

The following result extends [14, Theorem 3.1] to the non-additive setting.

**Proposition 4.3.** *Let $m: \mathcal{W} \to R$ be a $BDV$ set function, and $R \subset \mathcal{C}_\infty(\Omega)$, where $\Omega$ is as in Theorem 3.2. Then the set function $m_\omega := m(\cdot)(\omega)$ is real-valued, $BDV$ and $(s)$-bounded for every $\omega \in \Omega$. Moreover $m$ is $(s)$-bounded on $\mathcal{W}$.*

**Proof.** By arguing analogously as at the beginning of the proof of Proposition 3.9, we get that for every $\omega \in \Omega$ the set function $m_\omega$ defined by $m_\omega(A) := m(A)(\omega)$, $A \in \mathcal{W}$, is real-valued. For every $\omega \in \Omega$ we have

$$\overline{m_\omega}(G) = \sup_I\Big(\sum_{i\in I}(m(D_i)(\omega))\Big) = \sup_I\Big(\big(\sum_{i\in I} m(D_i)\big)(\omega)\Big) \le$$

$$\le \Big(\bigvee_I\big(\sum_{i\in I} m(D_i)\big)\Big)(\omega) = (\overline{m}(G))(\omega) \in \mathbb{R}, \tag{5}$$

since the pointwise supremum is less or equal than the corresponding lattice supremum in $\mathcal{C}(\Omega)$. Thus, $m_\omega$ is $BDV$ for each $\omega \in \Omega$. By [36, Theorem 3.2], for any disjoint sequence $(H_n)_n$ in $\mathcal{W}$ and for every $\omega \in \Omega$ we get $\lim_n \overline{m_\omega}(H_n) = 0$ and a fortiori $\lim_n m_\omega(H_n) = 0$. Thus we obtain the first part of the assertion.

Fix now any disjoint sequence $(H_n)_n$ in $\mathcal{W}$. By Theorem 3.2 there is a meager set $N_*$, depending on $(H_n)_n$, with

$$\Big[\bigwedge_{n=1}^{\infty}\Big(\bigvee_{s=n}^{\infty} m(H_s)\Big)\Big](\omega) = \inf_n\Big(\sup_{s\ge n}(m(H_s)(\omega))\Big) = \sup_n\Big(\inf_{s\ge n}(m(H_s)(\omega))\Big) = \Big[\bigvee_{n=1}^{\infty}\Big(\bigwedge_{s=n}^{\infty} m(H_s)\Big)\Big](\omega)$$

for every $\omega \in \Omega \setminus N_*$. From this we obtain $[(O)\lim_n m(H_n)](\omega) = 0$ for each $\omega \in \Omega \setminus N_*$. Since the complement of every meager subset of $\Omega$ is dense in $\Omega$, we have $[(O)\lim_n m(H_n)](\omega) = 0$ for every $\omega \in \Omega$, namely $(O)\lim_n m(H_n) = 0$. By arbitrariness of the chosen sequence $(H_n)_n$, we get $(s)$-boundedness of $m$ on $\mathcal{W}$. This ends the proof. □

**Remark 4.4.** Our theory here treated includes also the finitely additive case. Indeed, if $m: \mathcal{W} \to R$ is any finitely additive positive measure and $\{D_i: i \in I\}$ is any finite disjoint family of subsets of $G$, whose union we denote by $B$, then, by finite additivity and monotonicity, we get

$$\sum_{i\in I} m(D_i) = m\Big(\bigcup_{i\in I} D_i\Big) = m(B) \le m(G) \tag{6}$$

(see also [29, Proposition 3.4]). Thus, $m$ is $BDV$. □

We now deal with extensions of $k$-subadditive capacities, taking valued in *any* Dedekind complete lattice group. An algebra $\mathcal{W} \subset \mathcal{P}(G)$ is said to be *perfect* iff every monotone sequence $(A_n)_n$ of sets from $\mathcal{W}$, such that its limit in the set-theoretic sense belongs to $\mathcal{W}$, is eventually constant.

Academic Open Mathematics Research Journal
Vol. 1, No. 1, January 2016, pp. 1-13
Available online at http://www.akrpub.com/Journals.php

We now give the following

**Theorem 4.5.** *Let $R$ be any Dedekind complete lattice group, $\mathcal{W}$ be a perfect algebra, $m_0 : \mathcal{W} \to R$ be a BDV $k$-subadditive capacity. Then there is a (unique) BDV $k$-subadditive capacity $\widetilde{m} : \sigma(\mathcal{W}) \to R$, extending $m_0$, continuous from above and from below on $\sigma(\mathcal{W})$.*

**Proof.** First of all note that, by Proposition 3.9, for every $\omega \in \Omega$ the set function $m_\omega$ defined by $m_\omega(A) := m_0(A)(\omega)$, $A \in \mathcal{W}$, is a real-valued $k$-subadditive capacity, which is continuous from above and from below, because $\mathcal{W}$ is perfect. Since $m_0$ is $BDV$, then by Proposition 4.3 the set function $m_\omega$ is $BDV$ and $(s)$-bounded on $\mathcal{W}$ for every $\omega \in \Omega$. Since $m_\omega$ is $k$-subadditive, then it is *uniformly autocontinuous*, namely for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every $A$, $B \in \mathcal{W}$ with $m_\omega(B) \le \delta$ we get $m_\omega(A \cup B) \le m_\omega(A) + \varepsilon$ (see also [20,29]). (Indeed, it is enough to take, in correspondence with $\varepsilon > 0$, $\delta(\varepsilon) := \frac{\varepsilon}{k}$). By [20, Theorem 18], for each $\omega \in \Omega$ there is a real-valued capacity $\nu_\omega$, continuous from above and from below, defined on $\sigma(\mathcal{W})$, which is an extension of $m_\omega$. We claim that $\nu_\omega$ is $k$-subadditive on $\sigma(\mathcal{W})$. First of all, observe that $\nu_\omega$ is $k$-subadditive on $\mathcal{W}^+$, where $\mathcal{W}^+$ is the family of all subsets of $G$ which can be expressed as countable union of (increasing) sequences of elements of $\mathcal{W}$. Indeed, observe that, by construction, $\nu_\omega(E) = \lim_n m_\omega(E_n) = \sup_n m_\omega(E_n)$ whenever $E \in \mathcal{W}^+$, $E = \bigcup_{n=1}^\infty E_n$ (see also [20]). Let $A = \bigcup_{n=1}^\infty A_n$, $B = \bigcup_{n=1}^\infty B_n$, where $(A_n)_n$ and $(B_n)_n$ are two increasing sequences in $\mathcal{W}$. For each $n \in \mathbb{N}$, put $D_n := A_n \cup B_n$. Observe that $(D_n)_n$ is an increasing sequence in $\mathcal{W}$ and $\bigcup_{n=1}^\infty D_n = A \cup B$. By monotonicity and $k$-subadditivity of $m_\omega$ on $\mathcal{W}$ we get

$$m_\omega(D_n) \le m_\omega(A_n) + k\, m_\omega(B_n) \quad \text{for any } n \in \mathbb{N},$$
$$\nu_\omega(A \cup B) = \sup_n m_\omega(D_n) \le \sup_n m_\omega(A_n) + k \sup_n m_\omega(B_n) = \nu_\omega(A) + k\, \nu_\omega(B).$$

Moreover, $\nu_\omega$ is $k$-subadditive on $\mathcal{W}^* := \{A \subset G: \text{there is } D \in \mathcal{W}^+ \text{ with } D \supset A\}$: from this the claim will follow, since $\sigma(\mathcal{W}) \subset \mathcal{W}^*$. Indeed, by construction we have $\nu_\omega(A) = \inf\{\nu_\omega(D): D \supset A, D \in \mathcal{W}^+\}$ for any $A \in \mathcal{W}^*$ and $\omega \in \Omega$ (see also [20]). Choose arbitrarily $A_1$, $A_2 \in \mathcal{W}^*$. Pick $D_1$, $D_2 \in \mathcal{W}^+$, with $D_i \supset A_i$, $i = 1,2$. From monotonicity and $k$-subadditivity of $\nu_\omega$ on $\mathcal{W}^+$ we get

$$\nu_\omega(A_1 \cup A_2) \le \nu_\omega(D_1 \cup D_2) \le \nu_\omega(D_1) + k\, \nu_\omega(D_2).$$

Taking the infima with respect to $D_1$ and $D_2$, we get $\nu_\omega(A_1 \cup A_2) \le \nu_\omega(A_1) + k\, \nu_\omega(A_2)$.

Now let $\mathcal{A} := \{A \in \sigma(\mathcal{W})$: there is a function $f = f_A \in \mathcal{C}(\Omega)$ such that the set $\{\omega \in \Omega: \nu_\omega(A) \ne f_A(\omega)\}$ is meager in $\Omega\}$. Note that, by construction, $\mathcal{A}$ contains $\mathcal{W}$. We now claim that $\mathcal{A}$ is a monotone family. Let $(E_n)_n$ be any increasing sequence in $\mathcal{A}$, set $E_0 := \bigcup_{n=1}^\infty E_n$, and for every $n \in \mathbb{N}$ let $N_n$ be a meager set associated with $E_n$, according to the definition of $\mathcal{A}$. Put $f = \bigvee_n f_{E_n}$, where the involved supremum is taken in $\mathcal{C}(\Omega)$, and let $N_0 \subset \Omega$ be a meager set with $f(\omega) = \sup_n (f_{E_n}(\omega))$ for each $\omega \in \Omega \setminus N_0$, where the involved supremum is the pointwise one, according to Theorem 3.2. For every $\omega \in \Omega \setminus (\bigcup_{n=0}^\infty N_n)$ we have

$$\nu_\omega(E_0) = \sup_n \nu_\omega(E_n) = \sup_n [f_{E_n}(\omega)] = f(\omega).$$

Therefore $\mathcal{A}$ is closed under unions of increasing sequences. Analogously it is possible to see that $\mathcal{A}$ is closed under intersections of decreasing sequences. Hence, $\mathcal{A}$ is a monotone family containing $\mathcal{W}$ and contained in $\sigma(\mathcal{W})$, and therefore $\mathcal{A} = \sigma(\mathcal{W})$. Now, with the same notations as in Theorem 3.2, for each $A \in \sigma(\mathcal{W})$ put $\widetilde{m}(A) := a_A$ if $\widehat{a_A} = f_A$.

We get that $\widetilde{m}(A)$ is a $k$-subadditive capacity, continuous from above and from below (and hence $(s)$-bounded), which extends $m$. Note that the properties of $\widetilde{m}$ are consequences of the corresponding ones of $\nu_\omega$, $\omega \in \Omega$ and of the fact that $\mathcal{A}$ coincides with $\sigma(\mathcal{W})$. Here we prove only continuity from above, since the proof of the other properties is analogous. Let $A_0 = \bigcup_{n=1}^{\infty} A_n$, with $A_n \in \sigma(\mathcal{W})$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N} \cup \{0\}$, let $N_n^{'} := \{\omega \in \Omega: \nu_\omega(A_n) \neq f_{A_n}(\omega)\}$. Then $N_n^{'}$ is meager, and the set $N^{'} := \bigcup_{n=0}^{\infty} N_n^{'}$ is meager too. Thanks to continuity from below of $\nu_\omega$ on $\sigma(\mathcal{W})$, for each $\omega \in \Omega \setminus N^{'}$ we get

$$f_{A_0}(\omega) = \nu_\omega(A_0) = \sup_n \ \nu_\omega(A_n) = \sup_n [f_{A_n}(\omega)] = (\bigvee_n f_{A_n})(\omega).$$

Hence, we get $f_{A_0} = \bigvee_n f_{A_n}$. Continuity from below of $\widetilde{m}$ follows from this and arbitrariness of the chosen sequence $(A_n)_n$.

Now we prove that $\widetilde{m}$ is $BDV$ on $\sigma(\mathcal{W})$. Choose arbitrarily a finite disjoint family $\{D_i: i = 1,2,\dots,n\}$, with $D_i \in \sigma(\mathcal{W})$ for each $i$. There is a meager set $N \subset \Omega$, depending on $(D_i)_i$, with

$$(\sum_{i=1}^{n} \widetilde{m}(D_i))(\omega) = \sum_{i=1}^{n} \nu_\omega(D_i)$$

for each $\omega \in \Omega \setminus N$. For every $\varepsilon > 0$, $\omega \in \Omega \setminus N$ and $i = 1,\dots,n$ there is a set $F_i^{(\omega)} \in \mathcal{W}$ with $\nu_\omega(D_i \triangle F_i^{(\omega)}) \leq \frac{\varepsilon}{k\, 2^i}$ (see also [7,20]). From this, monotonicity and $k$-subadditivity of $\nu_\omega$ it follows that

$$(\sum_{i=1}^{n} \widetilde{m}(D_i))(\omega) = \sum_{i=1}^{n} \nu_\omega(D_i) \leq k \sum_{i=1}^{n} \nu_\omega(D_i \triangle F_i^{(\omega)}) + \sum_{i=1}^{n} m_\omega(F_i^{(\omega)}) \leq$$
$$\leq k \sum_{i=1}^{\infty} \frac{\varepsilon}{k\, 2^i} + \overline{m_\omega}(G) \leq \overline{m_0}(G)(\omega) + \varepsilon$$

for every $\omega \in \Omega \setminus N$. By arbitrariness of $\varepsilon$, we get

$$(\sum_{i=1}^{n} \widetilde{m}(D_i))(\omega) \leq \overline{m_0}(G)(\omega)$$

for any $\omega \in \Omega \setminus N$. Since the complement of a meager set is dense in $\Omega$, we obtain

$$(\sum_{i=1}^{n} \widetilde{m}(D_i))(\omega) \leq \overline{m_0}(G)(\omega)$$

for every $\omega \in \Omega$, namely

$$\sum_{i=1}^{n} \widetilde{m}(D_i) \leq \overline{m_0}(G). \tag{7}$$

From (7) and arbitrariness of the family $\{D_i: i = 1,\dots,n\}$, passing to the supremum, we deduce that $\widetilde{m}$ is $BDV$ on $\sigma(\mathcal{W})$. □

**Example 4.6.** An example of an extension satisfying Theorem 4.5 is the so-called *Stone extension*. Given an algebra $\mathcal{W} \subset \mathcal{P}(G)$, there exists a compact and totally disconnected topological space $Q^*$ such that $\mathcal{W}$ is algebraically isomorphic to the algebra $\mathcal{Q}$ of all open-closed subsets of $Q^*$. We call such an isomorphism $\psi: \mathcal{W} \to \mathcal{Q}$ the *Stone* *isomorphism*. Given any $BDV$ $k$-subadditive capacity $m: \mathcal{W} \to R$, define $m_0: \mathcal{Q} \to R$ by setting $m_0(B) := m(\psi^{-1}(B))$ for every $B \in \mathcal{Q}$. It is well-known that $\mathcal{Q}$ is perfect (see also [12]). By Theorem 4.5, $m_0$ admits a (unique) $R$-valued extension $\widetilde{m}$, defined on $\sigma(\mathcal{Q})$, continuous from above and from below on $\sigma(\mathcal{Q})$. The set function $\widetilde{m}$ is called the *Stone extension* of $m$. Thus, Theorem 4.5 extends [9, Theorem 2.6 and Lemma 2.7] to the $k$-subadditive case.

## 5. Conclusion

We proved that every $k$-subadditive capacity of bounded disjoint variation with values in *any* Dedekind complete lattice group, defined on a perfect algebra, admits an extension defined on the generated $\sigma$ –algebra. We have substantially extended to non-additive case and lattice groups some results about extensions of set functions proved in [9, 20], showing that our context is a strict strengthening of the finitely additive case. Thanks to perfectness of the involved algebra, it is possible to give positive results about extension of set functions even if the involved lattice group is not weakly $\sigma$-distributive, using the Maeda-Ogasawara-Vulikh representation theorem and similar extension results, proved for real-valued non-additive set functions. It remains still an open question, to find other kinds of extension theorems for set functions when the hypothesis of bounded disjoint variation is dropped, and to investigate other cases, in which different kinds of variations of non-additive set functions are considered (see also [29,36,37]).

## Acknowledgements

(a)This paper is supported by G.N.A.M.P.A. and University of Perugia.

(b)The author would thank Prof. Z. Lipecki for his helpful suggestions.

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